On the Conceptual Bases of Pseudo-scalar and Parity Violation

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By analyzing the handedness transformation of vector product under space inversions, we demonstrate that the so-called pseudo-vector and pseudo-scalar are not "pseudo". This demonstration casts doubt on the parity violation theory that is based on the nonzero measurements of "pseudo-scalars".

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1. Handedness Transformation of Vector Product under Space Inversions

Space is three dimensional, so there are three space inversions: 1d space inversion, equivalent to a mirror reflection, is a discrete transformation; 2d space inversion, equivalent to a 180-degree rotation, is a continuous transformation; 3d space inversion, equivalent to a product of 1d and 2d space inversions, is also a discrete transformation.

In a right-handed Cartesian system with three coordinates $\mathbf{x_1}$, $\mathbf{x_2}$ and $\mathbf{x_3}$, the vector product, denoted by $\times$, of $\mathbf{x_1}$ and $\mathbf{x_2}$, is equal to $\mathbf{x_3}$: $\mathbf{x_1} \times \mathbf{x_2} = \mathbf{x_3}$. Making a 2d space inversion, we would get $(-\mathbf{x_1}) \times (-\mathbf{x_2}) = \mathbf{x_3}$, which is the same as $\mathbf{x_1} \times \mathbf{x_2} = \mathbf{x_3}$. Making a 1d or 3d space inversion, it seems we would get $\mathbf{x_1} \times \mathbf{x_2} = -\mathbf{x_3}$, which is not consistent with the original definition $\mathbf{x_1} \times \mathbf{x_2} = \mathbf{x_3}$. Something is missing here.

It is easy to visualize that a right-handed Cartesian system remains right-handed under 2d space inversion, but becomes left-handed under 1d or 3d space inversion. To reflect this fact mathematically, it is necessary to let the vector product change sign under 1d or 3d space inversion: $\times \to -\times$, to transform the right-handed system to the left-handed one, then we get $\mathbf{x_1}(-\times)\mathbf{x_2} = -\mathbf{x_3}$, consistent with $\mathbf{x_1} \times \mathbf{x_2} = \mathbf{x_3}$. Similarly it is necessary to let the vector product remain unchanged under 2d space inversion: $\times \to \times$, to keep the coordinate system right-handed.

To summarize the above illustrations, we list the handedness transformation of vector product under space inversions:

$$\times \text{ (right-handed)} \to - \times \text{ (left-handed)}, \quad \text{under 1d or 3d space inversion}$$

$$\times \text{ (right-handed)} \to \times \text{ (right-handed)}, \quad \text{under 2d space inversion}$$

Using Levi-Civita symbol to express vector product: $x_i x_j \epsilon_{ijk} = x_k$ where i,j,k are 1,2,3 in cyclic order, we have

$$\epsilon_{ijk} \text{ (right-handed)} \to - \epsilon_{ijk} \text{ (left-handed)}, \quad \text{under 1d or 3d space inversion}$$

$$\epsilon_{ijk} \text{ (right-handed)} \to \epsilon_{ijk} \text{ (right-handed)}, \quad \text{under 2d space inversion}$$

The bottom line is: the vector product transforms differently under the discrete and continuous space inversions, in accord with the handedness transformation of the Cartesian coordinate system.

2. Are Pseudo-vector & Pseudo-scalar "Pseudo"?

Space angular momentum is defined by the vector product of space displacement and momentum vectors: $\mathbf{l} = \mathbf{r} \times \mathbf{p}$. Usually making a 3d space inversion, it appears one would get: $\mathbf{l}' = (-\mathbf{r}) \times (-\mathbf{p}) = \mathbf{l}$, and call the angular momentum "pseudo-vector" for it does not change sign under the space inversion. Based on our discussions above, however, what one does here is not a 3d space inversion: "it is a 2d space inversion", namely a 180-degree continuous rotation around an axis perpendicular to the plane that $\mathbf{r}$ and $\mathbf{p}$ reside, since the coordinate system remains right-handed. One may justify this by drawing a diagram to perform such a 180-degree continuous rotation.

To correctly make a 3d discrete space inversion, one needs to change not only the directions of $\mathbf{r}$ and $\mathbf{p}$, but also the handedness of the whole coordinate system, by transforming the vector product $\times$ to $-\times$: $\mathbf{l}' = (-\mathbf{r})(-\times)(-\mathbf{p}) = -\mathbf{l}$. Hence the space angular momentum is just a normal real vector, changing sign under 3d space inversion.

Spin, considered as part of total angular momentum, is also a real vector. If spin is right-handed in one coordinate system, then it becomes left-handed in its 3d or 1d space-inversed coordinate system. Namely, it flips direction: $\mathbf{s}' = -\mathbf{s}$ under 3d or 1d space inversion. In general, all angular momentums and vector products of any two vectors: $\mathbf{v_1} \times \mathbf{v_2}$, are real vectors.

A pseudo-scalar is usually defined by the scalar product of a pseudo-vector and a real vector, and is supposed to change sign under 3d space inversion. One example is $(\mathbf{v_1} \times \mathbf{v_2}) \cdot \mathbf{v_3}$. As mentioned above, $\mathbf{v_1} \times \mathbf{v_2}$ is not pseudo. So this scalar product, representing the total volume of a parallelepiped, is not pseudo: it is a well-defined real scalar in mathematics.

Another example is $\mathbf{s} \cdot \mathbf{p}$, where $\mathbf{s}$ is spin and $\mathbf{p}$ space momentum. Helicity, defined by the unit scalar product of spin and space momentum: $(\mathbf{s} \cdot \mathbf{p})/|\mathbf{s} \cdot \mathbf{p}|$, belongs to this category. Now that spin is a real vector, this type of scalar product is also real scalar. One may construct many other "pseudo-scalars" by using "pseudo-vectors", for instance, $(\mathbf{v_1} \times \mathbf{v_2}) \cdot [(\mathbf{v_3} \times \mathbf{v_4}) \times \mathbf{v_5}]$. In any case, these scalar products are real scalars.

3. On Parity Violation

The parity violation theory is based on the nonzero measurements of pseudo-scalars [Ref.1]. Early experiments were primarily focused on the measurements of pseudo-scalars of the type: $\mathbf{s} \cdot \mathbf{p}$, in weak interactions [Refs.2,3,4]. Recent attempts are also extended to the measurements of pseudo-scalars of the type: $(\mathbf{v_1} \times \mathbf{v_2}) \cdot \mathbf{v_3}$. Given the experimental results, we would wonder whether they truly imply parity violation. As shown above in §2, both types of scalar products are real scalars. From quantum mechanics, the nonzero measurements of real scalars do not imply parity violation.

In fact, these experiments can be explained by the strong correlation in weak interactions. In the β decay of Cobalt nuclei [Ref.2], the angular distribution of β decay is anisotropic if the nuclei are polarized as shown by the anisotropic distribution of γ decay; but becomes isotropic if the nuclei are unpolarized as indicated by the isotropic distribution of γ decay. This process shows a strong correlation between the β and γ decays. Similarly the anisotropic distribution of decaying electrons in meson/muon decay, is caused by the strong polarization of meson/muon beams [Ref.3]; and the neutrino helicity is determined by the circular polarization of resonant-scattered γ rays [Ref.4].

The principle of special relativity requires the equations of motion be Lorentz invariant, but not necessarily the outcome of the interactions, such as angular distribution, cross section or polarization. Furthermore, the symmetry principle applies only to an isolated system, while the decaying electrons on their own do not form an isolated system. The angular distribution or cross section of the decaying electrons, and the helicity of the decaying neutrinos, depend largely on several factors: initial conditions, external electromagnetic fields, and other participating particles including γ rays.

Ever since the discovery of parity violation in weak interactions, there have been many years in looking for symmetry breaking: P, CP, T, CPT, and even the whole Lorentz invariance violation. Unless one can establish a self-consistent symmetry-breaking theory without much parameter fixing, one should always give reasonable doubt about what one is looking for, and make sure there is no other reason than symmetry breaking.

As demonstrated above, the counterintuitive concepts like pseudo-vector and pseudo-scalar, widely used in physics but seldom in mathematics, are incorrect due to a lack of consideration of the handedness transformation of vector product under space inversions. Without getting into a philosophical debate on whether or why there should exist the Lorentz invariance in the first place, at least we can say: it is questionable to base a theory on the measurements of pseudo-scalars that are, as a matter of fact, real scalars.

References

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